Polymorphisms of reflexive digraphs

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Graphs and homomorphisms

Definition

- A **digraph** is a pair $\mathbb{G} = (G; \rightarrow)$, where G is the set of vertices and $\rightarrow \subseteq G^2$ is the set of edges.
- A **relational structure** is a tuple $\mathbb{G} = (G; E_1, \ldots, E_k)$, where G is the underlying set and $E_i \subseteq G^{n_i}$ is an n_i-ary relation.

Definition

A **homomorphism** from a digraph $\mathbb{G}$ to $\mathbb{H}$ is a map $f : G \rightarrow H$ that preserves edges

$$a \rightarrow b \text{ in } \mathbb{G} \quad \Longrightarrow \quad f(a) \rightarrow f(b) \text{ in } \mathbb{H}.$$

We write $\mathbb{G} \rightarrow \mathbb{H}$ if there exists a homomorphism from $\mathbb{G}$ to $\mathbb{H}$.

Constraint satisfaction problem (CSP)

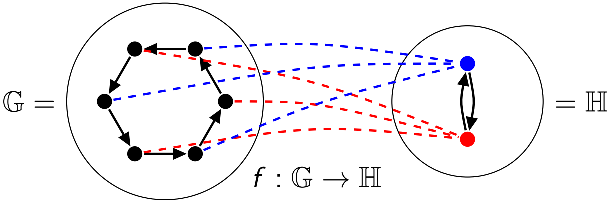
Definition

For a finite relational structure $\mathbb{H}$ we define

$$\mathrm{CSP}(\mathbb{H}) = \{\, \mathbb{G} \mid \mathbb{G} \to \mathbb{H} \,\}.$$

Example

- CSP() is the class of three-colorable graphs.
- CSP() is the class of bipartite graphs.



The computational complexity of CSP

The membership problem for $\mathrm{CSP}(\mathbb{H})$

- always decidable in nondeterministic polynomial time (**NP**)
- is decidable in polynomial time (**P**) for some $\mathbb{H}$

Dichotomy Conjecture (Feder, Vardi, 1993)

For every finite structure $\mathbb{H}$ the membership problem for $\mathrm{CSP}(\mathbb{H})$ is in **P** or **NP**-complete.

The dichotomy conjecture holds when $\mathbb{H}$

- is an undirected graph (Hell, Nešetřil), or
- has at most 3 elements (Bulatov), or
- a smooth directed graph (Barto, Kozik, Niven).

Open for directed graphs.

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Open for directed graphs.

Example: solving a system of equations

$$(\exists x, y, z \in \mathbf{Z}_5)(x + y = z \ \wedge\ x + x = y \ \wedge\ z = 1)$$

$$\Updownarrow$$

$$(\exists x, y, z \in \mathbf{Z}_5)((x, y, z) \in F_1 \ \wedge\ (x, x, y) \in F_1 \ \wedge\ z \in F_2),$$

where $F_1 = \{\, (x, y, z) \in \mathbf{Z}_5^3 : x + y = z \,\}$ and $F_2 = \{1\}$.

$$\Updownarrow$$

$$(\exists f : \{1, 2, 3\} \to \mathbf{Z}_5)((f(1), f(2), f(3)) \in F_1 \ \wedge$$
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$$\exists f : \mathbb{G} \to \mathbb{H},$$

where $\mathbb{G} = (\{1, 2, 3\}; E_1, E_2)$, $\mathbb{H} = (\mathbf{Z}_5; F_1, F_2)$

$$E_1 = \{\, (1, 2, 3),\ (1, 1, 2) \,\},\ \ E_2 = \{3\}.$$

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$$\mathbb{G} \in \mathrm{CSP}(\mathbb{H})$$

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$$\mathbb{G} \in \mathrm{CSP}(\mathbb{H})$$

CSP REDUCTIONS: CORES

Lemma

If $\mathbb{H}_1 \leftrightarrow \mathbb{H}_2$, *then* $\mathrm{CSP}(\mathbb{H}_1) = \mathrm{CSP}(\mathbb{H}_2)$. *In particular, if* $r : \mathbb{H} \to \mathbb{H}$ *is a retraction* ($r^2 = r$), *then* $\mathrm{CSP}(\mathbb{H}) = \mathrm{CSP}(\mathbb{H}|_{r(H)})$.

Lemma

For every finite relational structure $\mathbb{H}_1$ *there exists* $\mathbb{H}_2$ *such that*

1. $\mathbb{H}_2$ *is a directed graph (with unary relations),*
2. $\mathbb{H}_2$ *is a* **core**, *i.e., every endomorphism is bijective,*
3. *every singleton unary relation* $\{a\}$ *is in* $\mathbb{H}_2$, *and*

$\mathrm{CSP}(\mathbb{H}_1)$ *is polynomial time equivalent to* $\mathrm{CSP}(\mathbb{H}_2)$.

CSP reductions: polymorphisms

Definition

A **polymorphism** of $\mathbb{H}$ is a homomorphism $p : \mathbb{H}^n \to \mathbb{H}$, that is an n-ary map that preserves edges

$$a_1 \to b_1, \ldots, a_n \to b_n \implies p(a_1, \ldots, a_n) \to p(b_1, \ldots, b_n).$$

$\mathrm{Pol}(\mathbb{H}) = \{ p \mid p : \mathbb{H}^n \to \mathbb{H} \}$ is the **clone of polymorphisms**.

Lemma

If $\mathbb{H}_1$, $\mathbb{H}_2$ *have the same underlying set and* $\mathrm{Pol}(\mathbb{H}_1) \subseteq \mathrm{Pol}(\mathbb{H}_2)$, *then* $\mathrm{CSP}(\mathbb{H}_2)$ *is polynomial time reducible to* $\mathrm{CSP}(\mathbb{H}_1)$.

Question

Which polymorphisms guarantee that $\mathrm{CSP}(\mathbb{H})$ is in **P**?

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Which polymorphisms guarantee that $\mathrm{CSP}(\mathbb{H})$ is in **P**?

Nice polymorphisms

Theorem

$\mathrm{CSP}(\mathbb{H})$ is in **P** *if* $\mathrm{Pol}(\mathbb{H})$ *contains one of the following:*

- *a semilattice operation* (Jevons et. al.)

$$x \wedge y \approx y \wedge x, \quad x \wedge (y \wedge z) \approx (x \wedge y) \wedge z, \quad x \wedge x \approx x.$$

- *a near-unanimity operation*

$$p(y, x, \dots, x) \approx p(x, y, x, \dots, x) \approx \dots \approx p(x, \dots, x, y) \approx x,$$

- *a totally symmetric idempotent operation* (Dalmau, Pearson),

$$\{x_1, \dots, x_n\} = \{y_1, \dots, y_n\} \implies p(x_1, \dots, x_n) \approx p(y_1, \dots, y_n)$$

- *a Maltsev operation* (Bulatov, Dalmau)

$$p(x, y, y) \approx p(y, y, x) \approx x,$$

Nice polymorphisms cont.

Theorem

$\mathrm{CSP}(\mathbb{H})$ *is in* **P** *if* $\mathrm{Pol}(\mathbb{H})$ *contains one of the following:*

- *Edge operations* (Idziak, Marković, McKenzie, Valeriote, Willard)

$$\begin{aligned} p(y,y,x,x,\dots,x) &\approx x, \\ p(x,y,y,x,\dots,x) &\approx x, \\ p(x,x,x,y,\dots,x) &\approx x, \\ &\vdots \\ p(x,x,x,x,\dots,y) &\approx x. \end{aligned}$$

- *Jónsson operations* (Barto, Kozik),
- *Willard operations* (Barto, Kozik).

Weak near-unanimity

Theorem (McKenzie, Maróti)

For a locally finite variety $\mathcal{V}$ the followings are equivalent:

- $\mathcal{V}$ *omits type* **1** *(tame congruence theory),*
- $\mathcal{V}$ *has a Taylor term,*
- $\mathcal{V}$ *has a* **weak near-unanimity** *operation:*

$$p(y,x,\ldots,x) \approx \cdots \approx p(x,\ldots,x,y) \quad \text{and} \quad p(x,\ldots,x) \approx x.$$

Theorem (Bulatov, Larose, Zádori)

If $\mathbb{H}$ is a core and does not have a Taylor (or weak near-unanimity) polymorphism, then $\mathrm{CSP}(\mathbb{H})$ is **NP***-complete.*

Algebraic dichotomy conjecture

If $\mathbb{H}$ is a core and has a weak near-unanimity polymorphism, then $\mathrm{CSP}(\mathbb{H})$ is in **P**.

Algebraic results

Theorem (Barto)

If a finite relational structure has Jónsson polymorphisms

$$\begin{aligned} x &= d_0(x, y, z), \\ d_i(x, y, x) &= x \textit{ for all } i, \\ d_i(x, y, y) &= d_{i+1}(x, y, y) \textit{ for even } i, \\ d_i(x, x, y) &= d_{i+1}(x, x, y) \textit{ for odd } i, \\ d_n(x, y, z) &= z, \end{aligned}$$

then it has a near-unanimity polymorphism.

Valeriote's Conjecture

If a finite relational structure has Gumm polymorphisms, then it has an edge polymorphism.

Algebraic results for digraphs

Theorem (Larose, Zádori)

If a finite poset has Gumm polymorphisms

$$\begin{aligned} x &= d_0(x,y,z), \\ d_i(x,y,x) &= x \text{ for all } i, \\ d_i(x,y,y) &= d_{i+1}(x,y,y) \text{ for even } i, \\ d_i(x,x,y) &= d_{i+1}(x,x,y) \text{ for odd } i, \\ d_n(x,y,y) &= p(x,y,y), \text{ and} \\ p(x,x,y) &= y, \end{aligned}$$

then it has a near-unanimity polymorphism.

Theorem (Kun, Szabó)

There is a polynomial algorithm for checking if a poset has a near-unanimity polymorphism.

Algebraic results for digraphs

Theorem (Larose, Loten, Zádori)

If a finite symmetric reflexive digraph has Gumm polymorphisms, then it has a near-unanimity polymorphism.

Theorem (Kazda)

If a finite digraph has Maltsev polymorphism

$$p(y,x,x) \approx p(x,x,y) \approx y$$

then it has majority polymorphism

$$m(x,x,y) \approx m(x,y,x) \approx m(y,x,x) \approx x.$$

Conjecture

If a finite digraph has Gumm polymorphisms, then it has an near-unanimity polymorphism.

Main Result

Theorem

If a finite reflexive digraph $\mathbb{G}$ has Gumm polymorphisms

$$\begin{aligned} x &= d_0(x,y,z), \\ d_i(x,y,x) &= x \text{ for all } i, \\ d_i(x,y,y) &= d_{i+1}(x,y,y) \text{ for even } i, \\ d_i(x,x,y) &= d_{i+1}(x,x,y) \text{ for odd } i, \\ d_n(x,y,y) &= p(x,y,y), \text{ and} \\ p(x,x,y) &= y, \end{aligned}$$

then it has Jónsson polymorphisms (same as above, but $p(x,y,y) \approx y$).

Connectedness

Definition

A digraph $\mathbb{G}$ is **connected** if for all $a, b \in G$ there exists a path

$$a = a_0 \to a_1 \leftarrow a_2 \to a_3 \leftarrow \cdots \to a_n = b$$

with some pattern.

$\mathbb{G}$ is **strongly connected** if for all $a, b \in G$ there exist paths

$$a = a_0 \to a_1 \to a_2 \to a_3 \to \cdots \to a_n = b,$$
$$a = b_0 \leftarrow b_1 \leftarrow b_2 \leftarrow b_3 \leftarrow \cdots \leftarrow b_n = b.$$

$\mathbb{G}$ is **extremely connected** if for all $a, b \in G$ there exist a path

$$a = a_0 \leftrightarrow a_1 \leftrightarrow a_2 \leftrightarrow a_3 \leftrightarrow \cdots \leftrightarrow a_n = b.$$

Structure on $\mathbb{G}^{\mathbb{H}}$

Definition

Let $\mathbb{G}$, $\mathbb{H}$ be digraphs and $f, g \in H^G$ be maps. We write $f \to g$ iff

$$a \to b \text{ in } \mathbb{G} \implies f(a) \to g(b) \text{ in } \mathbb{H}.$$

Lemma

- *The set of homomorphisms from $\mathbb{G}$ to $\mathbb{H}$ is*

$$\mathbb{H}^{\mathbb{G}} = \{\, f \in H^G \mid f \to f \,\}.$$

- *If G is reflexive, then the Cartesian power of $\mathbb{G}$ is*

$$\mathbb{G}^n = \mathbb{G}^{\{\circlearrowleft\ \circlearrowleft\ \cdots\ \circlearrowleft\}}.$$

- *If $f \to g$ in $\mathbb{H}^{\mathbb{G}^n}$ and $f_1 \to g_1, \ldots, f_n \to g_n$ in $\mathbb{G}^{\mathbb{F}}$, then*

$$f(f_1, \ldots, f_n) \to g(g_1, \ldots g_n) \text{ in } \mathbb{H}^{\mathbb{F}}.$$

Connectedness on $\mathbb{G}^{\mathbb{G}}$

Theorem

Let $\mathbb{G}$ be a finite reflexive digraph admitting Gumm operations. If $\mathbb{G}$ is [strongly, extremely] connected, then so is $\mathbb{G}^{\mathbb{G}}$.

Proof.

- Take a minimal counterexample $\mathbb{G}$.
- $\{\mathrm{id}\}$ is a [strong, extreme] component of $\mathbb{G}^{\mathbb{G}}$.
- If $\mathbb{G}$ admits a ternary operation d satisfying
  - $d(x, y, y) \approx x$, or
  - $d(x, y, x) \approx d(x, x, y) \approx x$,

  then $d(x, y, z)$ is the first projection.
- Use the Gumm identities (or Hobby-McKenzie operations for omitting types **1** and **5**) to show that $\mathbb{G}$ satisfies $x \approx y$. □

Connectedness on $\mathbb{I}_2(\mathbb{G})$

Lemma

Let $\mathbb{G}$ be a finite reflexive digraph admitting Gumm operations. If $\mathbb{G}$ is [strongly, extremely] connected, then so is the digraph

$$\mathbb{I}_2(\mathbb{G}) = \{ f \in \mathbb{G}^{\mathbb{G}^2} \mid f(x,x) \approx x \}.$$

of idempotent binary polymorphisms of $\mathbb{G}$.

Definition

A digraph $\mathbb{K} \leq \mathbb{G}^{\mathbb{H}}$ is a **idempotent $\mathbb{G}$-subalgebra**, if it is closed under the idempotent polymorphisms of $\mathbb{G}$.

Corollary

Let $\mathbb{G}$ be a finite reflexive digraph admitting Gumm operations. If $\mathbb{G}$ is strongly connected, then it is extremely connected.

Unrefinable edges

Definition

An edge $f \to g$ in $\mathbb{G}^{\mathbb{G}}$ is **refinable** if there exists $h \in \mathbb{G}^{\mathbb{G}}$ such that

- $f \to h \to g$ and $f \neq h \neq g$, and
- $h(x) \in \{f(x), g(x)\}$ for all $x \in G$.

Lemma

If $f \to g$ is non-refinable, then $[[f \neq g]]$ is strongly connected.

Proof.

1. Take a nonconstant $\varphi : [[f \neq g]] \to \mathbb{G}^{\mathbb{G}}|_{\{f,g\}}$ homomorphism.
2. Look at the refinement $f \to h \to g$ where

$$h(x) = \begin{cases} (\varphi(x))(x), & \text{if } x \in [[f \neq g]] \\ f(x), & \text{otherwise.} \end{cases}$$

□

Getting Jónsson operations

Theorem

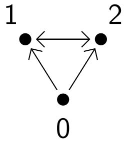
If a finite reflexive digraph $\mathbb{G}$ has Gumm polymorphisms, then it has Jónsson polymorphisms.

Proof.

- The digraph $\{ f \in \mathbb{G}^{3^{\mathbb{G}^3}} \mid f(x,y,x) = (x,y,x) \}$ is connected.
- Connect id with $s(x,y,z) = (z,y,z)$ via non-refinable links.
- We want a connection $\mathrm{id} = f_0, f_1, \ldots, f_n = s$ such that $x = \pi_1(f_0), \pi_1(f_1), \ldots, \pi_1(f_n) = z$ are Jónnson operations.
- Bad link: $[[f \neq g]]$ contains both (a,a,b) and (c,d,d), then $[[f \neq g]] \subseteq C^3$ for some strongly connected component C.
- C is extremely connected, so we can replace the bad (f,g) link with $f = h_0 \leftrightarrow h_1 \leftrightarrow \cdots \leftrightarrow h_n = g$.
- Refine all these links and we have no more bad links. □

Comments

- Let $\mathbb{G}$ be a poset, or a reflexive symmetric digraph. If $f \to g$ is a non-refinable edge in $\mathbb{G}^{\mathbb{G}}$, then $|[[f \neq g]]| = 1$.
- This is not true for reflexive digraphs:



- Dismantability is defined via "one-point elementary retractions".

Theorem

If a finite symmetric digraph $\mathbb{G}$ admits Gumm operations, then it admits Jónsson operations.

Decidability of NU

Theorem

If a finite reflexive digraph $\mathbb{G}$ admits a sequence of Jónsson operations, then it has one with length at most $16 \cdot |G|^7$.

Corollary

Given a finite reflexive digraph, it is decidable in polynomial time if it admits a near-unanimity operation.

Proof.

Use CSP and the bounded width algorithm. □

Theorem (Maróti)

*Given a finite algebra **A**, it is decidable if it has a near-unanimity term.*

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Totally symmetric operations

Definition

A digraph has the **fixed clique property**, if every endomorphism preserves some clique of the digraph.

Lemma

Every finite connected reflexive digraph that admits a near-unanimity operation has the fixed clique property.

Lemma

Every finite reflexive digraph that admits a near-unanimity operation also admits cyclic idempotent operations of all arities.

Theorem

Every finite reflexive digraph that admits an NU operation also admits totally symmetric idempotent operations of all arities.

Thank you!